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An inequality between intrinsic and extrinsic scalar curvature invariants for codimension 2 embeddings

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Abstract

We study the local and isometric embedding of an m-dimensional Lorentzian manifold in an $(m+2)$-dimensional pseudo-Euclidean space. An inequality is proven between the basic curvature invariants, i.e. the intrinsic scalar curvature and the extrinsic mean and scalar normal curvature. The inequality becomes an equality if the two components of the second fundamental form have a specified form with respect to some orthonormal basis of the manifold. As an application we look at the space–times embedded in a six-dimensional pseudo-Euclidean space for which the equality holds. They turn out to be Petrov type D models filled with an anisotropic perfect fluid and containing a timelike two-surface of constant curvature.

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1. Introduction

As early as 1873, a few years after the publication of the famous lecture of Riemann on the hypotheses which lie at the foundation of geometry, it was conjectured by Schläfli [8] that

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any m-dimensional Riemannian manifold could be locally and isometrically embedded in a d-dimensional Euclidean space with $d = m(m+1)/2$. This was later proven by Janet and Cartan and extended to manifolds with indefinite metric by Friedman [9]. In particular, any space–time can be locally and isometrically embedded in a pseudo-Euclidean space of at most 10 dimensions. Embeddings of space–time in non-flat ambient spaces are considered in [5,6].

From a mathematical point of view the embedding procedure was used as a classification scheme based on the embedding class [11,15], besides the usual classifications based on groups of motions or the Petrov type, and as a method of finding new solutions in some simple cases. See for example Fronsdal's study of the complete Schwarzschild solution [10].

With the recent advances in higher-dimensional physics (e.g. the Randall–Sundrum scenario [19,20], the space–time–matter theory [24], etc.) the extra dimensions which appear in the embedding of a space–time seem to be more than just some mathematical curiosity. In for example [1] isometric embeddings of several brane solutions of string theory are studied in a flat space with two time directions.

As expressed by Yau [25], a major problem in the study of embeddings of a space–time is the lack of *control of the extrinsic quantities of the embedded space in relation to the intrinsic quantities*. Recent work [2,3,13], originated by Chen, deals with this problem by obtaining optimal general inequalities between intrinsic and extrinsic curvature invariants. In [7] it was conjectured that if $\phi : M^m \to N^n(c)$ is an isometric immersion of a Riemannian manifold in a real space-form, then at every point p of M:

$$\|H\|^2 \geq \rho + \rho^{\perp} - c,$$

with $\|H\|^2$ the squared mean curvature, ρ the normalized scalar curvature and $\rho^{\perp}$ the scalar normal curvature. This conjecture was proved for $m = 2$ and $n \geq 4$ in [12] and in the case m is general and $n = m + 2$ in [7].

In this paper we will extend the above inequality to a Lorentzian manifold embedded locally and isometrically in a pseudo-Euclidean space with codimension 2. If equality holds the second fundamental form has a specific form and in Section 4 we show that space–times which realize the equality are Petrov type D anisotropic fluid models with a timelike two-surface of constant curvature. Remark that this approach is different from the methods usually used in the literature. There the form of the second fundamental form is determined by restrictions on the physical properties [21] or the Petrov type [15,23]. We on the other hand use a geometrical equality between intrinsic and extrinsic invariants of the space–time to determine the second fundamental form.

2. Definitions

In the following Greek indices are space–time indices and capital Latin letters denote normal space indices. Space–time indices are raised and lowered using $g^{\alpha\beta}$ and $g_{\alpha\beta}$.

Let M be an m-dimensional, time-orientable, Lorentzian manifold and E^{m+2} an $(m+2)$-dimensional pseudo-Euclidean space of signature $(+, \cdots, +, -, \varepsilon_{m+1}, \varepsilon_{m+2})$ with $\varepsilon_A = \pm 1$, $A = m+1, m+2$. We consider M to be locally and isometrically embedded in E^{m+2}.

The Levi-Civita connection on M is denoted by ∇ and on E^{m+2} by $\tilde{\nabla}$. The covariant derivative in E^{m+2} between two tangent vectors X and Y on M can be decomposed in a tangential and normal part:

$$\tilde{\nabla}_X Y = \nabla_X Y + A(X, Y),$$

with $A : TM \times TM \to N(M)$ the second fundamental form. If we choose an orthonormal basis $\{\xi_{m+1}, \xi_{m+2}\}$ in the normal space $N(M)$ of M, A is given by

$$A(X, Y) = \sum_{C=m+1}^{m+2} \varepsilon_C \eta(\tilde{\nabla}_X Y, \xi_C)\xi_C, \tag{1}$$

whereby η denotes the metric on E^{m+2}.

The integrability conditions for the existence of such an embedding are given by the Gauss–Codazzi–Ricci equations:

$$R_{\alpha\beta\gamma\mu} = \varepsilon_{m+1}(\Omega_{\alpha\gamma}\Omega_{\beta\mu} - \Omega_{\alpha\mu}\Omega_{\beta\gamma}) + \varepsilon_{m+2}(\Lambda_{\alpha\gamma}\Lambda_{\beta\mu} - \Lambda_{\alpha\mu}\Lambda_{\beta\gamma}), \tag{2}$$

$$\nabla_\gamma \Omega_{\alpha\beta} - \nabla_\beta \Omega_{\alpha\gamma} = \varepsilon_{m+2}(S_\beta \Lambda_{\alpha\gamma} - S_\gamma \Lambda_{\alpha\beta}), \tag{3}$$

$$\nabla_\gamma \Lambda_{\alpha\beta} - \nabla_\beta \Lambda_{\alpha\gamma} = \varepsilon_{m+1}(S_\gamma \Omega_{\alpha\beta} - S_\beta \Omega_{\alpha\gamma}), \tag{4}$$

$$\nabla_\beta S_\alpha - \nabla_\alpha S_\beta = \Omega_{\beta\gamma}\Lambda^\gamma_\alpha - \Omega_{\alpha\gamma}\Lambda^\gamma_\beta, \tag{5}$$

with $\Omega_{\alpha\beta}$ and $\Lambda_{\alpha\beta}$ the components of the second fundamental form and S_α the torsion vector. For an interpretation of this vector as a gauge field in a Kaluza–Klein view of embeddings see [16] or as a real connection on space–time see [17].

The mean curvature vector is defined as

$$\vec{H} = \frac{1}{m}(\varepsilon_{m+1}\Omega^\alpha_\alpha \xi_{m+1} + \varepsilon_{m+2}\Lambda^\alpha_\alpha \xi_{m+2})$$

and the normalized scalar curvature is

$$\rho = \frac{1}{m(m-1)} R = \frac{1}{m(m-1)} R^{\alpha\beta}_{\alpha\beta}.$$

Let $\{e_1, \ldots, e_{m-1}, e_m\}$ be an orthonormal basis of M. Because we have space–time applications in mind we take M time-orientable such that there exists a global, nowhere zero, timelike vector field which we denote with e_m. From (1) we have that

$$\Omega_{\alpha m} = -\eta(e_m, \tilde{\nabla}_{e_\alpha} \xi_{m+1}) = -\eta(e_\alpha, \tilde{\nabla}_{e_m} \xi_{m+1}), \tag{6}$$

with $\alpha = 1, \ldots, m-1$, and analogous relations hold for $\Lambda_{\alpha m}$. We can then introduce the following natural type of embeddings.

Definition 2.1. An embedding $\phi : (M^m, g) \to (E^{m+2}, \eta)$ with $\varepsilon_{m+1} = \varepsilon_{m+2} = 1$ is called causal-type preserving iff w.r.t. some orthonormal basis $\{e_1, \ldots, e_m\}$, $\tilde{\nabla}_{e_\alpha} \xi_A$ is spacelike, $A = m+1, m+2$ and $\alpha = 1, \ldots, m-1$.

Definition 2.2. An embedding $\phi : (M^m, g) \to (E^{m+2}, \eta)$ with $\varepsilon_{m+1} = \varepsilon_{m+2} = -1$ is called causal-type preserving iff w.r.t. some orthonormal basis $\{e_1, \ldots, e_m\}$, $\tilde{\nabla}_{e_m}\xi_A$ is timelike, $A = m+1, m+2$.

From (6) we see that causal-type preserving embeddings have $\Omega_{\alpha m} = \Lambda_{\alpha m} = 0, \alpha = 1, \ldots, m-1$. Remark that if this holds for some orthonormal basis it immediately holds for every orthonormal basis because the timelike vector e_m is fixed. The only freedom remaining are orthogonal transformations in the spacelike part of M which preserve the decompositions of Ω and Λ.

We further need two types of norms. If $B = (b_{\alpha\beta})$ is an $m \times m$ matrix in a Lorentzian manifold, we introduce the analogue of the Euclidean norm as

$$\|B\|^2 = \sum_{\alpha,\beta=1}^{m} \varepsilon_{\alpha\beta}|b_{\alpha\beta}|^2,$$

with $\varepsilon_{\alpha\beta} = \varepsilon_\alpha\varepsilon_\beta$. If $\vec{v} = v_1\xi_{m+1} + v_2\xi_{m+2}$ is a vector in normal space, we define the norm:

$$\|v\|_\perp^2 = \varepsilon_{m+1}(v_1)^2 + \varepsilon_{m+2}(v_2)^2.$$

Using these definitions we define the scalar normal curvature as

$$\rho^\perp = \frac{\sqrt{2}}{m(m-1)}\|[\Omega, \Lambda]\|.$$

From the Ricci equation (5) we see that $\rho^\perp$ corresponds, up to a constant factor, to the square length of the normal curvature tensor. Thus, the normal connection of M is flat if and only if $\rho^\perp = 0$ and this is equivalent to the simultaneous diagonalizability of both components of the second fundamental form. For further details on the scalar normal curvature see [7,12].

In the proof of the theorem we will need the following generalization of a lemma from [4].

Lemma 2.1. *Let $X = (x_{\alpha\beta})$ and $Y = (y_{\alpha\beta})$ be two symmetric $m \times m$ matrices in an m-dimensional Lorentzian space, such that w.r.t. some orthonormal basis $\{e_1, \ldots, e_m\}$, $x_{\alpha m} = y_{\alpha m} = 0$, with $\alpha = 1, \ldots, m-1$. Then the following inequality holds*

$$\|[X, Y]\|^2 \leq 2\|X\|^2\|Y\|^2$$

and equality holds if and only if with respect to some orthonormal basis $\{v, w, e_3, \ldots, e_m\}$ of M the two matrices take the form:

$$X_{\alpha\beta} = 2\tau v_{(\alpha}w_{\beta)}, \quad Y = \mu v_\alpha v_\beta - \mu w_\alpha w_\beta,$$

with $v_\alpha v^\alpha = w_\alpha w^\alpha = 1$.

Proof. Choose an orthonormal basis in M such that Y is diagonal. Then, with $X = (x_{\alpha\beta})$ and $Y = \mathrm{diag}(y_1, \ldots, y_m)$, we have

$$\|[X, Y]\|^2 = \sum_{\alpha\neq\beta=1}^{m} \varepsilon_{\alpha\beta}x_{\alpha\beta}^2(y_\alpha - y_\beta)^2.$$

Using $(y_\alpha - y_\beta)^2 \leq 2(y_\alpha^2 + y_\beta^2)$ and the condition that $x_{\alpha m} = 0, \forall \alpha = 1, \dots, m-1$, we find

$$\|[X,Y]\|^2 \leq 2 \sum_{\alpha\neq\beta=1}^{m} \varepsilon_{\alpha\beta} x_{\alpha\beta}^2 (y_\alpha^2 + y_\beta^2) \leq 2 \left(\sum_{\alpha,\beta=1}^{m} \varepsilon_{\alpha\beta} x_{\alpha\beta}^2\right)\left(\sum_{\gamma=1}^{m} y_\gamma^2\right) = 2\|X\|^2\|Y\|^2.$$

This proofs the inequality. There is equality if first $x_{\alpha\alpha} = 0$ and $y_\alpha + y_\beta = 0$ when $x_{\alpha\beta} \neq 0$. Suppose $x_{12} \neq 0$. Then we have $y_1 = -y_2$. If the last inequality becomes an equality, $y_3 = \cdots = y_m = 0$, and because $Y \neq 0$, we have that $y_1 = -y_2 \neq 0$. Also due to the last inequality, $x_{\alpha\beta} = 0$ if $(\alpha, \beta) \neq (1, 2)$. □

3. A basic inequality

We can now formulate and proof an inequality between the basic scalar curvature invariants of a embedded manifold, i.e. the intrinsic scalar curvature and the extrinsic squared mean curvature and scalar normal curvature.

Theorem 3.1. *Let $\phi : M^m \to E^{m+2}$ be a causal-type preserving, local and isometric embedding of a Lorentzian manifold M^m in a pseudo-Euclidean space E^{m+2}. Then*

$$\|H\|_\perp^2 \geq \rho + \rho^\perp, \tag{7}$$

if $\varepsilon_{m+1} = \varepsilon_{m+2} = 1$, and

$$\|H\|_\perp^2 \leq \rho - \rho^\perp, \tag{8}$$

if $\varepsilon_{m+1} = \varepsilon_{m+2} = -1$.

Proof. We choose any orthonormal basis $\{e_1, \dots, e_m\}$ on M and an orthonormal basis $\{\xi_{m+1}, \xi_{m+2}\}$ on E^{m+2} such that ξ_{m+1} is parallel to the mean curvature vector. All components will be expressed with respect to these bases. We then find, after contraction of the Gauss equation:

$$R = \varepsilon_{m+1}\left(\sum_{\alpha=1}^{m} \varepsilon_\alpha \Omega_{\alpha\alpha}\right)^2 - \varepsilon_{m+1}\sum_{\alpha,\beta=1}^{m} \varepsilon_{\alpha\beta}(\Omega_{\alpha\beta})^2 - \varepsilon_{m+2}\sum_{\alpha,\beta=1}^{m} \varepsilon_{\alpha\beta}(\Lambda_{\alpha\beta})^2$$
$$= m^2\|H\|_\perp^2 - \varepsilon_{m+1}\|\Omega\|^2 - \varepsilon_{m+2}\|\Lambda\|^2 = m^2\|H\|_\perp^2 - \|A\|_\perp^2.$$

Define the traceless matrices:

$$b_{\alpha\beta}^{m+1} = \varepsilon_{\alpha\beta}\Omega_{\alpha\beta} - \frac{1}{m}\left(\sum_{\gamma=1}^{m} \varepsilon_\gamma \Omega_{\gamma\gamma}\right) g_{\alpha\beta}, \qquad b_{\alpha\beta}^{m+2} = \varepsilon_{\alpha\beta}\Lambda_{\alpha\beta},$$

such that

$$\|b^{m+1}\|^2 = \sum_{\alpha,\beta=1}^{m} \varepsilon_{\alpha\beta}(\Omega_{\alpha\beta})^2 - \frac{1}{m}\left(\sum_{\gamma=1}^{m} \varepsilon_\gamma \Omega_{\gamma\gamma}\right)^2$$

and an analogous expression holds for b^{m+2}. We then have,

$$\|b\|_\perp^2 = \varepsilon_{m+1}\|b^{m+1}\|^2 + \varepsilon_{m+2}\|b^{m+2}\|^2 = \|A\|_\perp^2 - m\|H\|_\perp^2,$$

or putting this in the Gauss equation and using the definition of the normalized scalar curvature

$$\rho - \|H\|_\perp^2 = -\frac{1}{m(m-1)}\|b\|_\perp^2.$$

If we take the square of both sides and use the previous lemma, we find

$$\begin{aligned}[m(m-1)]^2(\rho - \|H\|_\perp^2)^2 &= (\varepsilon_{m+1}\|b^{m+1}\|^2 + \varepsilon_{m+2}\|b^{m+2}\|^2)^2 \\ &\geq 4\varepsilon_{m+1}\varepsilon_{m+2}\|b^{m+1}\|^2\|b^{m+2}\|^2 \geq 2\varepsilon_{m+1}\varepsilon_{m+2}\|[b^{m+1}, b^{m+2}]\|^2,\end{aligned}$$

where the last inequality holds if $\varepsilon_{m+1} = \varepsilon_{m+2}$. We further have that $\|[b^{m+1}, b^{m+2}]\|^2 = \|[\Omega, \Lambda]\|^2$. Thus

$$(\rho - \|H\|_\perp^2)^2 \geq (\rho^\perp)^2.$$

If $\varepsilon_{m+1} = \varepsilon_{m+2} = +1$, then $\rho - \|H\|_\perp^2 \leq 0$ and

$$\|H\|_\perp^2 \geq \rho + \rho^\perp.$$

If $\varepsilon_{m+1} = \varepsilon_{m+2} = -1$, then $\rho - \|H\|_\perp^2 \geq 0$ and

$$\|H\|_\perp^2 \leq \rho - \rho^\perp.$$

This proves the theorem. □

Corollary 3.1. *There is equality in (7) or (8) if and only if with respect to an orthonormal basis on M the two components of the second fundamental form take the form*

$$\Omega_{\alpha\beta} = 2\tau v_{(\alpha} w_{\beta)} + \nu g_{\alpha\beta}, \tag{9}$$

$$\Lambda_{\alpha\beta} = \mu v_\alpha v_\beta - \mu w_\alpha w_\beta, \tag{10}$$

with $v_\alpha v^\alpha = w_\alpha w^\alpha = 1$ *and* $\mu \neq 0, \tau \neq 0$.

Proof. This follows from the equality in the lemma and the definition of the matrices b^{m+1} and b^{m+2}. □

4. Space–times realizing the equality

We now characterize those space–times for which the equality is satisfied in (7) or (8). We denote the orthonormal tetrad of the space–time M with $\{v^\alpha, w^\alpha, q^\alpha, u^\alpha\}$, $u_\alpha u^\alpha = -1$, and decompose the covariant derivatives of the basis vectors as in [22]:

$$\nabla_\beta u_\alpha = w_\alpha A_\beta + v_\alpha B_\beta + q_\alpha C_\beta, \qquad \nabla_\beta w_\alpha = u_\alpha A_\beta + v_\alpha D_\beta + q_\alpha E_\beta,$$
$$\nabla_\beta v_\alpha = u_\alpha B_\beta - w_\alpha D_\beta + q_\alpha F_\beta, \qquad \nabla_\beta q_\alpha = u_\alpha C_\beta - w_\alpha E_\beta - v_\alpha F_\beta.$$

The various space–time vectors are decomposed as

$$A_\alpha = A_v v_\alpha + A_w w_\alpha + A_q q_\alpha - A_u u_\alpha.$$

Because we know the specialized form of the components of the second fundamental form (9) and (10), we can integrate the Gauss–Codazzi–Ricci equations step by step.

4.1. The Codazzi equations

After projecting the Codazzi equations (3) and (4) on the various tetrad components, we find the relations

$$\nabla_u \nu = \nabla_q \nu = 0, \qquad \nabla_u \ln \mu = \nabla_u \ln \tau, \qquad \nabla_q \ln \mu = \nabla_q \ln \tau$$

and

$$A_\alpha = -\frac{\nabla_v \nu}{\tau} u_\alpha + A_v v_\alpha - \nabla_u \ln \tau w_\alpha, \qquad B_\alpha = -\frac{\nabla_w \nu}{\tau} u_\alpha - \nabla_u \ln \tau v_\alpha - A_v w_\alpha,$$
$$D_\alpha = \frac{1}{2} A_v u_\alpha + \frac{1}{2\tau}(\nabla_v \nu - \nabla_w \tau - \varepsilon\mu S_v) v_\alpha + \frac{1}{2\tau}(\nabla_v \tau - \nabla_w \nu - \varepsilon\mu S_w) w_\alpha + \frac{1}{2} E_v q_\alpha,$$
$$E_\alpha = E_v v_\alpha + \nabla_q \ln \tau\, w_\alpha + \frac{\nabla_v \nu}{\tau} q_\alpha, \qquad F_\alpha = \nabla_q \ln \tau v_\alpha - E_v w_\alpha + \frac{\nabla_w \nu}{\tau} q_\alpha.$$

The Codazzi equations put no restriction on C_α. The torsion vector satisfies $S_u = S_q = 0$ and

$$\varepsilon\nu S_v = -\frac{\mu \nabla_w \nu}{\tau}, \qquad \varepsilon \nu S_w = \frac{\mu \nabla_v \nu}{\tau}. \tag{11}$$

4.2. The Ricci equation

We then use (11) in the Ricci equation (5) and obtain a differential equation for ν:

$$\nabla_v \nabla_v \nu + \nabla_w \nabla_w \nu + \frac{2\tau \nabla_v \nu \nabla_w \nu}{\mu^2} - \frac{\tau \nabla_v \tau \nabla_v \nu}{\mu^2} - \frac{\tau \nabla_w \tau \nabla_w \nu}{\mu^2}$$
$$+ \nabla_v \ln\left(\frac{\mu}{\tau}\right) \nabla_v \nu + \nabla_w \ln\left(\frac{\mu}{\tau}\right) \nabla_w \nu - \frac{2\tau^2 - \mu^2}{\mu^2} D_v \nabla_w \nu$$
$$+ \frac{2\tau^2 - \mu^2}{\mu^2} D_w \nabla_v \nu = -2\varepsilon\nu\tau^2. \tag{12}$$

If $\nu = 0$ the Ricci equation gives differential equations for the unknowns S_v and S_w.

4.3. Subspaces of constant curvature

Let $p_\alpha^\beta = \delta_\alpha^\beta - v_\alpha v^\beta - w_\alpha w^\beta$ be the projection operator on the timelike two-surface T_2 orthogonal to the vectors v^α and w^α. The two components of the second fundamental form of the embedding of T_2 in the space–time M are

$$\Omega^v_{\alpha\beta} = p_{(\alpha}{}^\mu p^\gamma_{\beta)} \nabla_\mu v_\gamma = B_u p_{\alpha\beta}$$

and

$$\Omega^w_{\alpha\beta} = p_{(\alpha}{}^\mu p^\gamma_{\beta)} \nabla_\mu w_\gamma = A_u p_{\alpha\beta}.$$

Using the Gauss equation we find the Riemann tensor of the surface T_2:

$$\begin{aligned} {}^2R_{\alpha\beta\gamma\delta} &= p^\mu_\alpha p^\sigma_\beta p^\rho_\gamma p^\upsilon_\delta R_{\mu\sigma\rho\upsilon} + 2\Omega^v{}_{\alpha[\gamma}\Omega^v_{\delta]\beta} + 2\Omega^w{}_{\alpha[\gamma}\Omega^w_{\delta]\beta}, \\ &= 2\{\varepsilon\nu^2 + A_u^2 + B_u^2\} p_{\alpha[\gamma} p_{\delta]\beta}. \end{aligned} \tag{13}$$

With $\nabla_u \nu = \nabla_q \nu = 0$ and

$$A_u = \frac{\nabla_v \nu}{\tau}, \qquad B_u = \frac{\nabla_w \nu}{\tau},$$

it is a small calculation to show that the coefficient in (13) has zero derivative in the u- and q-direction. Therefore, the space–times embedded in a six-dimensional pseudo-Euclidean space such that the equality is realized in (7) or (8) contain a timelike two-surface of constant curvature and hence admit a group G_3 of motions whose orbits are T_2. We further have the following theorem from [21].

Theorem 4.1. *Space–times admitting a group G_3 of motions acting on non-null orbits V_2 are of Petrov type D or O.*

Because we have that for example, $v^\alpha w^\beta v^\gamma w^\delta C_{\alpha\beta\gamma\delta} = (1/3)\varepsilon(\tau^2 + \mu^2) \neq 0$, the embedded space–times must be of Petrov type D.

Let us further denote with $h_\alpha^\beta = \delta_\alpha^\beta + u_\alpha u^\beta - q_\alpha q^\beta$ the projection operator on the spacelike two-surface S_2 orthogonal to u^α and q^α. We can again find the two components of the second fundamental form for the embedding of S_2 in M:

$$\Omega^u_{\alpha\beta} = A_w h_{\alpha\beta} \text{ and } \Omega^q_{\alpha\beta} = -E_w h_{\alpha\beta}.$$

The Riemann tensor of the spacelike surface S_2 is given by

$${}^2R_{\alpha\beta\gamma\delta} = 2\{\varepsilon(\nu^2 - \tau^2 - \mu^2) + A_w^2 + E_w^2\} h_{\alpha[\gamma} h_{\delta]\beta}.$$

We find that the space–time is reducible [18], i.e. contains two two-surfaces of constant curvature, if and only if

$$\nabla_v\{\varepsilon(\nu^2 - \tau^2 - \mu^2) + (\nabla_u \ln \tau)^2 + (\nabla_q \ln \tau)^2\} = 0$$

and

$$\nabla_w\{\varepsilon(\nu^2 - \tau^2 - \mu^2) + (\nabla_u \ln \tau)^2 + (\nabla_q \ln \tau)^2\} = 0.$$

4.4. The matter content

From the Gauss equation (2) we can find the energy-momentum tensor of the embedded space–time as $\kappa T_{\alpha\beta} = R_{\alpha\beta} - (1/2)Rg_{\alpha\beta}$ or

$$\kappa T_{\alpha\beta} = \varepsilon\{(\tau^2 + \mu^2 - 3\nu^2)g_{\alpha\beta} + 4\nu\tau v_{(\alpha}w_{\beta)} - (\tau^2 + \mu^2)v_\alpha v_\beta - (\tau^2 + \mu^2)w_\alpha w_\beta\}. \tag{14}$$

With respect to the orthonormal tetrad $\{v^\alpha, w^\alpha, q^\alpha, u^\alpha\}$ the energy-momentum tensor describes a zero-flux imperfect fluid. After a diagonalization in the spacelike two-space S_2 we find the energy density ρ and anisotropic pressures p_i,

$$\rho = \varepsilon(3\nu^2 - \tau^2 - \mu^2), \qquad p_1 = -3\varepsilon\nu^2 - 2\varepsilon\nu\tau,$$
$$p_2 = -3\varepsilon\nu^2 + 2\varepsilon\nu\tau, \qquad p_3 = -\rho.$$

If we want any of the three energy conditions to be satisfied (see [14, p. 88]), the extra embedding dimensions must always be timelike, $\varepsilon = -1$, and in each of the three cases some inequalities must hold. For example, the strong energy condition is satisfied if and only if $\varepsilon = -1$,

$$\tau^2 + \mu^2 - 2\tau\nu \geq 0 \quad \text{and} \quad \tau^2 + \mu^2 + 2\tau\nu \geq 0.$$

Remark that a vacuum space–time or non-null Einstein–Maxwell fields are not possible because $\tau \neq 0$ and $\mu \neq 0$.

4.5. Coordinate representation of the metric

Because the metric has a timelike two-surface of constant curvature we can find coordinates such that the metric reads

$$\mathrm{d}s^2 = \mathrm{e}^{2\lambda(y,z)}\,\mathrm{d}y^2 + \mathrm{e}^{2\gamma(y,z)}\,\mathrm{d}z^2 + Y^2(y,z)\{\mathrm{d}x^2 - \Sigma^2(x,k)\,\mathrm{d}t^2\}, \tag{15}$$

with $\Sigma(x,k) = (\sin(x), x, \sinh(x))$ for $k = (1,0,-1)$ [21]. By means of coordinate transformations which leave the form of the metric invariant, we can put $Y(y,z) = y$ if $\partial_\alpha Y \neq 0$. Working with these so-called canonical coordinates and the orthonormal tetrad

$$\vec{v} = \mathrm{e}^{-\lambda}\partial_y, \qquad \vec{w} = \mathrm{e}^{-\gamma}\partial_z, \qquad \vec{q} = y^{-1}\partial_x, \qquad \vec{u} = -y^{-1}\Sigma^{-1}\partial_t,$$

the Codazzi equations give

$$\nabla_v \nu = 0, \qquad S_w = 0, \qquad \varepsilon\nu S_v = -\frac{\mu}{y}\mathrm{e}^{-\lambda} \neq 0, \quad \text{thus } \nu(z) \neq 0 \text{ and } S_v \neq 0.$$

We find that the embedding is non-minimal ($\nu \neq 0$). From the derivatives of τ and μ resulting from the Codazzi equations we find that

$$\partial_y(\gamma + \lambda) = 0 \quad \text{and} \quad \partial_z\partial_y(\gamma - \lambda) = 0,$$

or $\lambda = \lambda_1(y) + \lambda_2(z)$ and $\gamma = -\lambda_1(y) + \gamma_2(z)$. By means of a coordinate transformation $e^{\gamma_2(z)}\,dz \to dz$, we can put $\gamma_2(z) = 0$. Further integration of the Codazzi equations gives

$$\tau = y\exp(f_1(z) + 2\lambda_1(y)) \quad \text{and} \quad \mu = y\exp(f_2(z) + 2\lambda_1(y)),$$

with

$$\partial_z f_1(z) = -2\partial_z\lambda_2(z) + \frac{\exp(2f_2(z) - f_1(z) - \lambda_2(z))}{\nu(z)}$$

and

$$\partial_z f_2(z) = -2\partial_z\lambda_2(z) + \frac{\exp(f_1(z) - \lambda_2(z))}{\nu(z)}.$$

We also have

$$\partial_z\nu(z) = \exp(f_1(z) - \lambda_2(z))$$

and

$$S_v = -\varepsilon\frac{\exp(f_2(z) - \lambda_2(z) + \lambda_1(y))}{\nu(z)}.$$

Using the expression for the torsion vector in the Ricci equation (12) we find

$$\partial_z \exp(-\lambda_2(z)) = -\varepsilon C\nu(z)\exp(f_1(z)) \tag{16}$$

and $\exp(2\lambda_1(y)) = Cy^{-2}$, with C a constant. Thus, given ν, τ and μ the metric reads

$$ds^2 = Cy^{-2}\,e^{2\lambda_2(z)}\,dy^2 + C^{-1}y^2\,dz^2 + y^2\{dx^2 - \Sigma^2(x,k)\,dt^2\}, \tag{17}$$

with λ_2 a solution of (16) and $\tau = Cy^{-1}\exp(f_1(z))$.

As mentioned above, this represents a Petrov type D anisotropic perfect fluid which is non-minimally embedded in a six-dimensional pseudo-Euclidean space such that equality holds in (8) with $\varepsilon = -1$.

On the other hand, if $Y = a =$ constant, the Codazzi equations with the above tetrad give $A_\alpha = B_\alpha = E_\alpha = F_\alpha = 0$. From the Ricci identity $2\nabla_{[\gamma}\nabla_{\beta]}u_\alpha = u^\sigma R_{\sigma\alpha\beta\gamma}$ we find that $\nu = 0$ or the embedding must be minimal. Using the metric (8) and the Gauss equation we see that

$$\partial_x^2\Sigma(x,k) = 0.$$

This can only occur if $k = 0$ and $\Sigma(x,k) = x$. The Ricci tensor has Segré type [(11)(1, 1)] with eigenvalues

$$\begin{aligned}\xi_1 = \xi_2 = \varepsilon(\tau^2 + \mu^2) &= \partial_y^2\gamma\,e^{-2\lambda} + \partial_z^2\lambda\,e^{-2\gamma} + (\partial_y\gamma)^2\,e^{-2\lambda} + (\partial_z\lambda)^2\,e^{-2\gamma}\\ &\quad - \partial_y\gamma\partial_y\lambda\,e^{-2\lambda} - \partial_z\gamma\partial_z\lambda\,e^{-2\gamma} \neq 0\end{aligned} \tag{18}$$

and $\xi_3 = \xi_4 = 0$. Remark that because $\tau \neq 0, \mu \neq 0$ this metric is no non-null Einstein–Maxwell field (because $R \neq 0$) but an anisotropic fluid with $\rho = p_3 = 0$ and

$p_1 = p_2$. This fluid satisfies the weak and strong energy conditions if and only if $\varepsilon = -1$, but cannot satisfy the dominant energy condition.

The metric reads

$$\mathrm{d}s^2 = \mathrm{e}^{2\lambda(y,z)}\,\mathrm{d}y^2 + \mathrm{e}^{2\gamma(y,z)}\,\mathrm{d}z^2 + a^2\{\mathrm{d}x^2 - x^2\,\mathrm{d}t^2\}, \tag{19}$$

with λ and γ satisfying (18).

5. Summary

We showed that space–times for which the equality is realized in (7) or (8) contain a timelike two-surface of constant curvature and are of Petrov type D. Two classes appear. Either the models are anisotropic perfect fluid models which are non-minimally embedded in a six-dimensional pseudo-Euclidean space with signature $(3(+), 3(-))$ and have coordinate representation (17) or the anisotropic models satisfying the weak or strong energy conditions are minimally embedded and have coordinate representation (19).

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